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Two Doubly Infinite Sets of Series for π

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It is shown how two doubly infinite sets of series involving π may be obtained using closure and a particular set of orthonormal functions. The first set of series converge to π from below, while the second set converge to π from above. The leading terms of the simplest of these series are alternately lower and upper bounds to, and successive truncations of Wallis's formula. Some convergence properties of these series are examined. © 1990 Academic Press, Inc.

Numerous expressions for π appear in the literature. Recently [1], for instance, π was related to the ratio: the logarithm of the product to the logarithm of the least common multiple of Fibonacci numbers. Expressions for π in terms of infinite series are also quite common, some of these based on trigonometric identities. In the present paper, π is shown to be expressible in terms of two, two parameter families of infinite series.

We derive, in what follows, the equalities

$$\pi = \frac{1}{2n+l+\frac{3}{2}} \sum_{n'=0}^{\infty} (A_{n'}^{(n,l)})^2, \tag{1}$$

$$\pi = \sum_{n'=0}^{\infty} A_{n'}^{(n,l)} B_{n'}^{(n,l)}, \tag{2}$$

where

$$A_{n'}^{(n,l)} = \frac{1}{2^{n+n'-2l-1}} \sqrt{\frac{(2n+2l+1)!\,(2n'+2l+1)!}{(n+l)!\,(n'+l)!\;n!\;n'!}} \sum_{p=0}^{n} \sum_{p'=0}^{n'} 2^{2(p+p')} \times \frac{\Gamma(-n+p)\Gamma(-n'+p')(l+p+p'+1)!\,(l+p')!\,(l+p)!}{\Gamma(-n)\Gamma(-n')\,p!\,p'!\,(2l+2p+1)!\,(2l+2p'+1)!}, \tag{3}$$

$$B_{n'}^{(n,l)} = \frac{1}{2^{n+n'-2l-1}} \sqrt{\frac{(2n+2l+1)!\,(2n'+2l+1)!}{(n+l)!\,(n'+l)!\;n!\;n'!}} \sum_{p=0}^{n} \sum_{p'=0}^{n'} 2^{2(p+p')} \times \frac{\Gamma(-n+p)\Gamma(-n'+p')(l+p+p')!\,(l+p')!\,(l+p)!}{\Gamma(-n)\Gamma(-n')\,p!\,p'!\,(2l+2p+1)!\,(2l+2p'+1)!}. \tag{4}$$

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TABLE I

Series for π Obtained from Equality (1)

n \ l	0	1	2
0	$\pi = \frac{8}{3} + \frac{4}{9} + \frac{1}{45} + \frac{1}{210} + \frac{5}{3024} + \cdots$	$\pi = \frac{128}{45} + \frac{54}{225} + \frac{16}{1575} + \frac{8}{4725} + \cdots$	$\pi = \frac{512}{175} + \frac{256}{1225} + \frac{64}{11025} + \frac{32}{40425} + \cdots$
1	$\pi = \frac{18}{7} + \frac{5}{14} + \frac{4}{21} + \frac{1}{60} + \cdots$	$\pi = \frac{5408}{2025} + \frac{4232}{14175} + \frac{64}{405} + \frac{484}{42525} + \cdots$	
2	$\pi = \frac{225}{88} + \frac{175}{528} + \frac{5}{22} + \frac{1}{165} + \cdots$		

TABLE II

Series for π Obtained from Equality (2)

n \ l	0	1	2	3
0	$\pi = 4 - \frac{2}{3} - \frac{1}{10} - \frac{1}{28} - \cdots$	$\pi = \frac{32}{9} - \frac{16}{45} - \frac{4}{105} - \frac{1}{189} - \cdots$	$\pi = \frac{256}{75} - \frac{128}{525} - \frac{32}{1575} - \frac{16}{3465} - \cdots$	$\pi = \frac{4096}{1225} - \frac{2048}{11025} - \frac{512}{40425} - \frac{256}{105105} - \cdots$
1	$\pi = 5 - \frac{11}{12} - \frac{2}{3} - \cdots$			

In expressions (1) and (2), n, l are independent of each other and can have any integral value $\geqslant 0$. Thus, one has different series for each choice of n, l. Some of these series are listed in Tables I and II. The simplest series to obtain, see Tables I and II, are the fastest converging and hence the most interesting, namely, when $n=0$. In this case, (3) and (4) reduce to single sums. Moreover, if one uses Legendre's duplication formula [2] to transform $(2n'+2l+1)!$, $(2l+2p'+1)!$ one can rewrite these single sums as:

$$A_{n'}^{(0,l)}=\frac{(l+1)!}{\Gamma(l+\frac{3}{2})}\sqrt{\frac{l!\,\Gamma(n'+l+\frac{3}{2})\,2^{2l+1}\pi^{1/2}}{(2l+1)!\,n'!}}\times {}_2F_1(-n',l+2;l+\tfrac{3}{2};1),$$

$$B_{n'}^{(0,l)}=\frac{l!}{\Gamma(l+\frac{3}{2})}\sqrt{\frac{l!\,\Gamma(n'+l+\frac{3}{2})\,2^{2l+1}\pi^{1/2}}{(2l+1)!\,n'!}}\times {}_2F_1(-n',l+1;l+\tfrac{3}{2};1).$$

The hypergeometric functions ${}_2F_1$ with unit argument in these expressions can be summed using Gauss's summation theorem [3], yielding the closed forms:

$$A_{n'}^{(0,l)}=\frac{2^{2l+1}l!(l+1)!\,\Gamma(n'-\frac{1}{2})}{(2l+1)!\,\Gamma(-\frac{1}{2})}\sqrt{\frac{\Gamma(l+\frac{3}{2})}{\Gamma(n'+l+\frac{3}{2})\,n'!}},\tag{5a}$$

$$B_{n'}^{(0,l)}=\frac{2^{2l+1}(l!)^2\,\Gamma(n'+\frac{1}{2})}{(2l+1)!\,\Gamma(\frac{1}{2})}\sqrt{\frac{\Gamma(l+\frac{3}{2})}{\Gamma(n'+l+\frac{3}{2})\,n'!}}.\tag{5b}$$

Hence, from (1), with $n=0$:

$$\begin{aligned}\pi&=\frac{1}{(l+\frac{3}{2})}\left(\frac{2^{2l+1}l!(l+1)!}{(2l+1)!}\right)^2\times\sum_{n'=0}^{\infty}\left(\frac{\Gamma(n'-\frac{1}{2})}{\Gamma(-\frac{1}{2})}\right)^2\frac{\Gamma(l+\frac{3}{2})}{\Gamma(n'+l+\frac{3}{2})\,n'!}\\&=\frac{1}{(l+\frac{3}{2})}\left(\frac{2^{2l+1}l!(l+1)!}{(2l+1)!}\right)^2\times\left(1+\frac{\Gamma(l+\frac{3}{2})}{\Gamma(-\frac{1}{2})^2}\sum_{n'=1}^{\infty}\frac{\Gamma(n'-\frac{1}{2})^2}{\Gamma(n'+l+\frac{3}{2})\,n'!}\right),\end{aligned}\tag{6a}$$

while from (2),

$$\pi = \frac{l!}{(l+1)!}\left(\frac{2^{2l+1}l!(l+1)!}{(2l+1)!}\right)^2 \times \sum_{n'=0}^{\infty} \frac{\Gamma(n'-\frac{1}{2})\,\Gamma(n'+\frac{1}{2})\,\Gamma(l+\frac{3}{2})}{\Gamma(-\frac{1}{2})\,\Gamma(\frac{1}{2})\,\Gamma(n'+l+\frac{3}{2})\,n'!}$$

$$= \frac{l!}{(l+1)!}\left(\frac{2^{2l+1}l!(l+1)!}{(2l+1)!}\right)^2 \times \left(1 - \frac{\Gamma(l+\frac{3}{2})}{2\Gamma(\frac{1}{2})^2}\sum_{n'=1}^{\infty}\frac{\Gamma(n'-\frac{1}{2})\,\Gamma(n'+\frac{1}{2})}{\Gamma(n'+l+\frac{3}{2})\,n'!}\right). \tag{6b}$$

If $n \neq 0$, summing (3) and (4) over p also yields single sum expressions. Thus

$$A_{n'}^{(n,l)} = \frac{(l+1)!}{\Gamma(l+\frac{3}{2})}\sqrt{\frac{l!\,\Gamma(n'+l+\frac{3}{2})\,2^{2l+1}\pi^{1/2}\Gamma(n+l+\frac{3}{2})}{(2l+1)!\,n'!\,\Gamma(l+\frac{3}{2})\,n!}} \times \sum_{p'=0}^{n'}\frac{\Gamma(-n'+p')\,\Gamma(l+2+p')\,\Gamma(l+\frac{3}{2})}{\Gamma(-n')\,\Gamma(l+2)\,\Gamma(l+\frac{3}{2}+p')\,p'!} \times {}_2F_1(-n, l+p'+2; l+\tfrac{3}{2}; 1).$$

Using Gauss's summation formula this may also be written in terms of an ${}_3F_2$, namely:

$$A_{n'}^{(n,l)} = \frac{(l+1)!(-1)^n\Gamma(\frac{3}{2})\,2^n}{\Gamma(l+\frac{3}{2})\,\Gamma(\frac{3}{2}-n)}\sqrt{\frac{(n+l)!\,\Gamma(n'+l+\frac{3}{2})\,2^{2l+1}\pi^{1/2}}{(2n+2l+1)!\,n'!\,n!}} \times {}_3F_2(-n', l+2, \tfrac{3}{2}; l+\tfrac{3}{2}, \tfrac{3}{2}-n; 1). \tag{7}$$

Unlike the corresponding expression when $n=0$, expression (7) cannot be summed in closed form except if $l=0$, in which case the ${}_3F_2$ reduces to an ${}_2F_1$ which can be summed by Gauss's formula, yielding

$$A_{n'}^{(n,0)} = (-1)^n\,2^n\sqrt{\frac{\Gamma(n'+\frac{3}{2})\,2\pi^{1/2}}{(2n+1)!\,n'!}}\,\frac{\Gamma(n'-n-\frac{1}{2})}{\Gamma(n'-n+\frac{3}{2})\,\Gamma(-n-\frac{1}{2})}. \tag{8}$$

One notes that (3) and (4) are meaningful (by taking limits) in spite of the poles $\Gamma(z)$ has at $0, -1, -2, \ldots$.

To obtain (1) and (2), one uses the complete set of three-dimensional orthonormal functions frequently encountered in mathematical physics [4]:

$$\phi_{nlm_l}(\mathbf{r}) = R_{nl}(r)\,Y_{m_l}^l(\theta, \phi). \tag{9}$$

In (9), the $Y_{m_l}^l(\theta, \phi)$ are the standard spherical harmonics [5],

$$R_{nl}(r)=\left\{\frac{2\Gamma(n+l+\frac{3}{2})}{n!}\right\}^{1/2}\frac{r^l e^{-r^2/2}}{\Gamma(l+\frac{3}{2})}\,{}_1F_1(-n;l+\tfrac{3}{2};r^2),$$

and $n=0, 1, 2, ..., l=0, 1, 2, ...$, while the integer m_l satisfies $|m_l|\leqslant l$ so that

$$\int d\mathbf{r}\;\phi^*_{nlm_l}(\mathbf{r})\,\phi_{n'l'm_{l'}}(\mathbf{r})=\delta_{nn'}\,\delta_{ll'}\,\delta_{m_lm_{l'}} \tag{10}$$

and

$$\sum_{n,l,m_l}\phi_{nlm_l}(\mathbf{r})\,\phi^*_{nlm_l}(\mathbf{r}')=\delta(\mathbf{r}-\mathbf{r}'). \tag{11}$$

This set arises in the three dimensional quantum-mechanical harmonic-oscillator eigenvalue problem, namely

$$\tfrac{1}{2}[-\nabla^2+r^2]\,\phi_{nlm_l}(\mathbf{r})=(2n+l+\tfrac{3}{2})\,\phi_{nlm_l}(\mathbf{r}).$$

According to the quantum mechanical Virial Theorem [6], when the potential energy (PE) depends on the second power of r ($\mathrm{PE}=r^2/2$), the expectation value of the potential energy equals the expectation value of the kinetic energy:

$$\int d\mathbf{r}\;\phi^*_{nlm_l}(\mathbf{r})\,\frac{r^2}{2}\,\phi_{nlm_l}(\mathbf{r})=\int d\mathbf{r}\;\phi^*_{nlm_l}(\mathbf{r})-\frac{\nabla^2}{2}\,\phi_{nlm_l}(\mathbf{r}).$$

Hence,

$$\int d\mathbf{r}\;\phi^*_{nlm_l}(\mathbf{r})\,1/2(-\nabla^2+r^2)\,\phi_{nlm_l}(\mathbf{r})=\int d\mathbf{r}\;\phi^*_{nlm_l}(\mathbf{r})\,r^2\phi_{nlm_l}(\mathbf{r})$$

$$=\int d\mathbf{r}\;\phi^*_{nlm_l}(\mathbf{r})(2n+l+\tfrac{3}{2})\,\phi_{nlm_l}(\mathbf{r}),$$

which, using (10) equals $2n+l+\frac{3}{2}$.

On the other hand, using the fact that $\phi_{nlm_l}(\mathbf{r})$ is a complete set, one can write, using (11)

$$\int d\mathbf{r}\;\phi^*_{nlm_l}(\mathbf{r})\,r^2\phi_{nlm_l}(\mathbf{r})$$

$$=\int d\mathbf{r}'\int d\mathbf{r}\;\phi^*_{nlm_l}(\mathbf{r})\,r\delta(\mathbf{r}-\mathbf{r}')\,r'\phi_{nlm_l}(\mathbf{r}')$$

$$=\int d\mathbf{r}'\int d\mathbf{r}\;\phi^*_{nlm_l}(\mathbf{r})r\sum_{n',l',m_{l'}}\phi_{n'l'm_{l'}}(\mathbf{r})\,\phi^*_{n'l'm_{l'}}(\mathbf{r}')\,r'\phi_{nlm_l}(\mathbf{r}')$$

$$=\sum_{n',l',m_{l'}}\left|\int d\mathbf{r}\;\phi^*_{nlm_l}(\mathbf{r})\,r\phi_{n'l'm_{l'}}(\mathbf{r})\right|^2.$$

Thus

$$(2n+l+\tfrac{3}{2})=\sum_{n',l',m_{l'}}\left|\int d\mathbf{r}\;\phi^*_{nlm_l}(\mathbf{r})\,r\phi_{n'l'm_{l'}}(\mathbf{r})\right|^2. \qquad (12)$$

Since r is a scalar, this implies l' must equal l, and $m_{l'}$ must equal m_l. Moreover, the angular integrals in (12) can be evaluated directly. One obtains

$$2n+l+\tfrac{3}{2}=\sum_{n'=0}^{\infty}\left\{\int_0^{\infty} r^2\,dr\,R_{nl}(r)\,rR_{n'l}(r)\right\}^2, \qquad (13)$$

using the fact that the R_{nl}'s are real.

On the right hand side of (13) each summand involves $1/\pi$. If one defines

$$\frac{1}{\sqrt{\pi}}A_{n'}^{(n,l)}\equiv\int_0^{\infty} r^2\,dr\,R_{nl}(r)\,rR_{n'l}(r),$$

the integral can be directly evaluated, yielding (3). One notes that all the sums in (1) are series of positive terms. Thus, terminating these series after a finite number of terms always gives a lower bound for π as opposed, for example, to Leibniz's formula [7]

$$\pi=4-\tfrac{4}{3}+\tfrac{4}{5}-\tfrac{4}{7}+\cdots.$$

It is interesting to examine the leading terms in the series (1), namely, those for which $n=n'$.

For $n=0$,

$$\pi\approx\frac{2}{2l+3}(A_0^{(0,l)})^2=\frac{2}{2l+3}\left(\frac{(l+1)!\,2^{l+1}}{(2l+1)!!}\right)^2. \qquad (14a)$$

$$\text{If } l=0,\ \pi\approx 2\times\frac{2\times 2}{1\times 3};\qquad \text{if}\quad l=1,\ \pi\approx 2\times\frac{2\times 2\times 4\times 4}{1\times 3\times 3\times 5};$$

$$\text{if}\quad l=2,\ \pi\approx\frac{2\times 2\times 4\times 4\times 6\times 6}{1\times 3\times 3\times 5\times 5\times 7},\ \text{etc.} \qquad (14b)$$

Letting $l\to\infty$, we obtain Wallis's result for π [8]

$$\pi=2\times\frac{2\times 2\times 4\times 4\times 6\times 6\times 8\times 8\cdots}{1\times 3\times 3\times 5\times 5\times 7\times 7\times 9\cdots}. \qquad (15)$$

This result is to be expected, since, as $l\to\infty$, the off-diagonal matrix elements $\int_0^{\infty} r^2\,drR_{0l}(r)\,rR_{n'l}(r)$, $n'\neq 0$, tend to zero, because, for large l,

$$\int_0^{\infty} r^2\,dr\,R_{0l}(r)\,rR_{n'l}(r)\sim\int_0^{\infty} r^2\,drR_{0l}(r)\,R_{n'l}(r)=0$$

by (10). Moreover, as seen from (14b), if $l=0, 1, 2 \ldots$, one obtains Wallis's expression truncated after an *even* number of terms in the numerator and denominator.

For $n=1, 2$, the leading terms ($n'=n$) in the series (1) are the expressions in (14a) multiplied by numerical factors which $\to 1$ as $l \to \infty$, namely,

$$\begin{aligned} &\frac{2}{2l+3}(A_0^{(0,l)})^2 \frac{(2l+4.5)^2}{(2l+3)(2l+7)}, \\ &\frac{2}{2l+3}(A_0^{(0,l)})^2 \frac{(4l^2+22l+225/8)^2}{(2l+3)(2l+5)^2(2l+11)}, \end{aligned} \tag{16}$$

respectively. Thus, as $l \to \infty$, they reduce to these expressions. In other words, just as for $n=0$, so also for $n=1, 2$ (and one expects, for any n), the leading term by itself $\to \pi$, as $l \to \infty$ indicating the series converge fast for large l.

In an exactly analogous way to the steps leading to (12), one notes from (10) and (11), that

$$\begin{aligned} &\int d\mathbf{r}\, \phi^*_{nlm_l}(\mathbf{r})\, \phi_{nlm_l}(\mathbf{r}) \\ &= 1 = \int d\mathbf{r}\, \phi^*_{nlm_l}(\mathbf{r})\, r \frac{1}{r} \phi_{nlm_l}(\mathbf{r}) \\ &= \int d\mathbf{r}' \int d\mathbf{r}\, \phi^*_{nlm_l}(\mathbf{r})\, r\delta(\mathbf{r}-\mathbf{r}') \frac{1}{r'} \phi_{nlm_l}(\mathbf{r}') \\ &= \int d\mathbf{r}' \int d\mathbf{r}\, \phi^*_{nlm}(\mathbf{r})\, r \sum_{n',l',m_l'} \phi_{n'l'm_l'}(\mathbf{r})\, \phi^*_{n'l'm_l'}(\mathbf{r}') \frac{1}{r'} \phi_{nlm_l}(\mathbf{r}') \\ &= \sum_{n',l',m_l'} \int d\mathbf{r}\, \phi^*_{nlm_l}(\mathbf{r})\, r\phi_{n'l'm_l'}(\mathbf{r}) \int d\mathbf{r}'\, \phi^*_{n'l'm_l'}(\mathbf{r}') \frac{1}{r'} \phi_{nlm_l}(\mathbf{r}'). \end{aligned}$$

Thus, as above, using the fact that r, $1/r$ are scalars, we obtain

$$1 = \sum_{n'} \int d\mathbf{r}\, \phi^*_{nlm_l}(\mathbf{r})\, r\phi_{n'lm_l}(\mathbf{r}) \int d\mathbf{r}'\, \phi_{n'lm_l}(\mathbf{r}') \frac{1}{r'} \phi_{nlm_l}(\mathbf{r}'), \tag{17}$$

and, integrating over the angular variables, we have

$$1 = \sum_{n'} \int_0^\infty r^2\, dr\, R_{nl}(r)\, rR_{n'l}(r) \int_0^\infty r'^2\, dr'\, R_{n'l}(r') \frac{1}{r'} R_{nl}(r') \tag{18}$$

which is just (2), if one defines

$$\frac{1}{\sqrt{\pi}} B_{n'}^{(n,l)} \equiv \int_0^\infty r^2\, dr\, R_{n'l}(r) \frac{1}{r} R_{nl}(r).$$

When this integral is evaluated directly, it yields (4).

The leading term of (6b) yields

$$\frac{\pi}{2} \approx \frac{2}{1}, \qquad \frac{2\times 2\times 4}{1\times 3\times 3}, \qquad \frac{2\times 2\times 4\times 4\times 6}{1\times 3\times 3\times 5\times 5}, \tag{19}$$

for $l = 0, 1, 2, \ldots$.

This is Wallis's formula (15) truncated after an *odd* number of terms in the numerator and denominator.

It is clear that the expressions in (19), unlike those in (14b) which are *lower* bounds, are all *upper* estimates for π, since, unlike series (6a) where all terms are positive, all the terms in the series (6b), other than the leading term, are negative.

Wallis's expression can thus be broken down into successively more accurate lower and upper bounds, a bit reminiscent of Archimides's [10] lower and upper bounds for π obtained by inscribing polygons in circles and also circumscribing these circles with polygons of successively more and more sides. Thus

$$\begin{gathered} \frac{2\times 2}{1\times 3} < \frac{\pi}{2} < \frac{2}{1}, \\ \frac{2\times 2\times 4\times 4}{1\times 3\times 3\times 5} < \frac{\pi}{2} < \frac{2\times 2\times 4}{1\times 3\times 3}, \\ \frac{2\times 2\times 4\times 4\times 6\times 6}{1\times 3\times 3\times 5\times 5\times 7} < \frac{\pi}{2} < \frac{2\times 2\times 4\times 4\times 6}{1\times 3\times 3\times 5\times 5}. \end{gathered} \tag{20}$$

It is easy to estimate the rate of convergence, which is quite remarkable for large l. Thus, for $n = 0$, probably the most useful series, one sees that, starting with the leading term ($n' = 0$) of the series (6a), the n'th term is obtained by multiplication by

$$\left(\frac{\Gamma(n' - \frac{1}{2})}{\Gamma(-\frac{1}{2})}\right)^2 \frac{\Gamma(l + \frac{3}{2})}{\Gamma(n' + l + \frac{3}{2})\, n'!}. \tag{21}$$

For $n' = 10$ and $l = 0$, this factor equals 2.32×10^{-5}, while for $n' = 10$ and $l = 10$, it is merely 3.35×10^{-10}.

For large values of n', one can easily obtain an estimate for the remainder in the series (6a) with the help of formulas derived from Stirling's asymptotic series, in particular the formula [9]

$$\lim_{z \to \infty} \frac{\Gamma(z+a)}{\Gamma(z+b)} = z^{a-b}.$$

The n'th term in (6a), namely,

$$C(l)\,\frac{\Gamma(n'-\tfrac{1}{2})\,\Gamma(n'-\tfrac{1}{2})}{\Gamma(n'+l+\tfrac{3}{2})\,\Gamma(n'+1)}, \tag{22}$$

where $C(l)$ is a constant depending on l, converges as $n' \to \infty$ to $C(l)n'^{-l-7/2}$.

Since the series (1) have positive terms only, we can estimate their remainder R beyond the Nth term by converting the remainder sum to an integral. In the case $n=0$ one obtains by (22), for large N,

$$R \approx C(l) \int_N^\infty dn'\, n'^{-l-7/2} = \frac{C(l)\,N^{-l-5/2}}{(l+\tfrac{5}{2})} = \frac{N \times N\text{th term}}{l+\tfrac{5}{2}}$$

$$\approx \frac{N}{l} \times (N\text{th term}). \tag{23}$$

One notes that, if $N > l$, the ratio in (23) becomes larger than the Nth term. There is, consequently, no advantage in including more than about l terms in the series approximation for π. One should, therefore, increase l and N simultaneously, thus benefiting from the faster convergence with increasing l, while keeping the ratio $N/l \leqslant 1$. The convergence properties of the series (6b) are similar to those of the series (6a).

Conclusion: Two (infinite) series converging to π are obtained for $n = 0, 1, 2, \ldots$, and $l = 0, 1, 2\ldots$. These series are all distinct; the first approaching π monotonically from below, while the second approach π monotonically from above. The leading terms of the first of these series (for $n = 0, 1, 2$) approach Wallis's expression for π as $l \to \infty$. For $n = 0$, the leading terms of the first series are Wallis's expression truncated after an even number of factors in the numerator and denominator, while for the second series, they are Wallis's expression truncated after an odd number of factors in the numerator and denominator. One thus obtains successively better upper and lower bounds for π. The new results (1), (2), (3), and (4) are obtained with the help of and important set of functions of mathematical physics. The series converge very fast with increasing l. To minimize the remainder upon truncating a particular series, it is best to simultaneously have large N and l, where N is the order of the partial sum taken.

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